

## GENERALIZED DEGREE DISTANCE OF TREES, UNICYCLIC AND BICYCLIC GRAPHS

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**ABSTRACT.** A generalization of degree distance of graphs we recently proposed as a new topological index. In this paper, the new index is studied in trees, in unicyclic graphs of girth  $k$  and in some special classes of bicyclic graphs. Lower-bound and upper-bound values and analytical formulae to calculate this index in the studied graphs are given.

**Keywords:** Generalized degree distance, unicyclic graphs, bicyclic graphs, trees.

### INTRODUCTION

A graph invariant is any function on a graph that does not depend on the labeling of its vertices. Topological indices TIs are graph invariants calculated on the graphs associated to molecules. The distance-based TIs have been widely used in theoretical chemistry to establish relations between the structure and the properties of molecules: correlations with physical, chemical and biological properties of chemical compounds have been reported [1]. In this paper, only simple graphs are considered.

Let  $G$  be a connected graph with vertex and edge sets  $V(G)$  and  $E(G)$ , respectively, their cardinalities being  $n=|V(G)|$  and  $m=|E(G)|$ . In a molecular graph, the vertices represent atoms and the edges the covalent bonds. The distance between the vertices  $u$  and  $v$  of  $G$  is denoted by  $d_G(u, v)$  ( $d(u, v)$  for short), and represent the length of a minimum path connecting them. Let  $d_G(v)$  be the degree of a vertex  $v$  or the valence of a given atom in the hydrogen depleted molecular graph. The eccentricity, denoted

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by  $\varepsilon(v)$ , is defined as the maximum distance from vertex  $v$  to any other vertex in  $G$ . The diameter of a graph,  $d(G)$ , is the maximum eccentricity over all vertices in  $G$ . For any vertex  $v \in V(G)$ , the open neighborhood of  $v$  is the set  $N(v) = \{u \in V(G) \mid uv \in E(G)\}$  and the closed neighborhood of  $v$  is the set  $N[v] = N(v) \cup \{v\}$ . The girth of a graph is the length of a shortest cycle contained in the graph. A connected graph  $G$  with  $n$  vertices and  $m$  edges is called unicyclic if  $m = n$ ;  $G$  is called bicyclic if  $m = n + 1$ .

The additively weighted Harary index was defined in [2] as follows:

$$H_A(G) = \sum_{\{u,v\} \subseteq V(G)} d^{-1}(u,v)(d_G(u) + d_G(v)).$$

Dobrynin and Kochetova in [3] and Gutman in [4] introduced a new graph invariant, called the degree distance. It is defined as:

$$D'(G) = \sum_{\{u,v\} \subseteq V(G)} d(u,v)(d_G(u) + d_G(v)).$$

The first Zagreb index was originally defined as [5]:

$$M_1(G) = \sum_{u \in V(G)} d_G(u)^2$$

The first Zagreb index can also be expressed as the sum over all the edges of  $G$ :

$$M_1(G) = \sum_{uv \in E(G)} [d_G(u) + d_G(v)].$$

We refer the readers to [6] for the proof of this fact and for more information on the Zagreb index.

The first Zagreb co-index of a graph  $G$  is defined in [7] as:

$$\overline{M}_1(G) = \sum_{uv \notin E(G)} [d_G(u) + d_G(v)].$$

Let  $d(G, k)$  be the number of pairs of vertices of a graph  $G$  located at distance  $k$ ,  $\lambda$  be a real number, and

$$W_\lambda(G) = \sum_{k \geq 1} d(G, k) k^\lambda$$

be a Wiener-type invariant of  $G$  associated to a real number  $\lambda$ , see [8, 9] for details.

A generalization of the degree distance, denoted by  $H_\lambda(G)$  was proposed [10]. For every vertex  $u$ , the “degree distance sum” is defined as:

$$H_\lambda(u) = D^\lambda(u) d_G(u)$$

where the distance sum is calculated on the distance matrix raised at power  $\lambda$ :

$$D^\lambda(u) = \sum_{v \in V(G)} d^\lambda(u, v).$$

So we have:

$$H_\lambda(G) = \sum_{u \in V(G)} H_\lambda(u) = \sum_{u \in V(G)} D^\lambda(u) d_G(u) = \sum_{\{u, v\} \subseteq V(G)} d^\lambda(u, v) (d_G(u) + d_G(v))$$

where  $\lambda$  is a real number. If  $\lambda = 0$ , then  $H_\lambda(G) = 4m$ . When  $\lambda = 1$ , this new topological index  $H_\lambda(G)$  equals the degree distance index (i.e. Dobrynin or Schultz index). The properties of the degree distance index were studied in [11-14]. Also, if  $\lambda = -1$ , then  $H_\lambda(G) = H_A(G)$  (see above). The relation of our new index with other intensely studied indices motivated our present (and future) study.

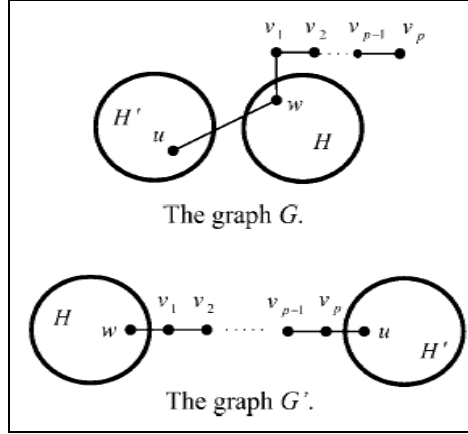
Throughout this paper,  $C_n$ ,  $P_n$ ,  $K_n$  and  $S_n$  denote the cycle, path, complete and star graphs on  $n$  vertices, respectively. The complement of a graph  $G$  is a graph  $H$  on the same vertices such that two vertices of  $H$  are adjacent if and only if they are not adjacent in  $G$ . The graph  $H$  is usually denoted by  $\overline{G}$ . Our other notations are standard and taken mainly from [1, 15, 16].

Extremal graph theory is a branch of the mathematical field of graph theory. Extremal graph theory studies extremal (maximal or minimal) graphs which satisfy a certain property. Extremality can be taken with respect to different graph invariants, such as order, size or girth. The problem of determining extremal values and corresponding extremal graphs of some graph invariants is the topic of several papers [11-14, 17-26]. In this paper, we characterize  $n$ -vertex unicyclic graphs with girth  $k$ , having minimum and maximum generalization degree distance and we derive the formula of this index for some special classes of bicyclic graphs. Furthermore, we determine the minimum and maximum of this index for trees.

## RESULTS AND DISCUSSION

Let us construct the graph  $G$  as follows: Let  $H$  and  $H'$  be two disjoint connected graphs such that  $|V(H)| \geq 1$  and  $|V(H')| \geq 1$ . Suppose that there is  $w \in V(H)$  such that  $d_H(w) \geq 2$ ,  $u \in V(H')$  and  $P = wv_1v_2 \dots v_p$  is a pendant path of length  $p \geq 1$  attached at  $w$ , and the edge  $wu$  is connecting  $H$  and  $H'$  (see Figure 1). Let  $G' = \pi(H, w, P, H')$  be the graph obtained from the graph  $G$  by removing the edge  $wu$  and inserting the edge  $v_pu$ .

We call such a transformation from  $G$  to  $G'$  a  $\pi$ -transform of a graph  $G$ . Note that if  $d_G(w) = 2$ , then  $G$  and  $G'$  are isomorphic [27].



**Figure 1.**  $\pi$ -transform applied to  $G$  at vertex  $w$ .

**Theorem 2.1.** Let  $G' = \pi(H, w, P, H')$  be a  $\pi$ -transform of a graph  $G$  and  $\lambda$  be a positive integer. Then  $H_\lambda(G') \geq H_\lambda(G)$ , and the equality is held if and only if  $d_G(w) = 2$ .

**Proof.** If  $d_G(w) = 2$ , then it is obvious the isomorphism  $G \cong G'$ , so assume that  $d_G(w) > 2$ . The only vertices that change degree after performing the  $\pi$ -transform are  $w$  and  $v_p$ . So

$$\begin{aligned}
 H_\lambda(w, G) &= d_G(w) \left( \sum_{v \in V(H)} d^\lambda(v, w) + (1^\lambda + 2^\lambda + \dots + p^\lambda) \right) \\
 &\quad + \sum_{v \in V(H')} (d(u, v) + 1)^\lambda, \\
 H_\lambda(w, G') &= (d_G(w) - 1) \left( \sum_{v \in V(H)} d^\lambda(v, w) + (1^\lambda + 2^\lambda + \dots + p^\lambda) \right) \\
 &\quad + \sum_{v \in V(H')} (d(u, v) + p + 1)^\lambda, \\
 H_\lambda(v_p, G) &= \left( \sum_{v \in V(H)} (d(v, w) + p)^\lambda + (1^\lambda + 2^\lambda + \dots + (p-1)^\lambda) \right) \\
 &\quad + \sum_{v \in V(H')} (d(u, v) + p + 1)^\lambda,
 \end{aligned}$$

$$H_\lambda(v_p, G') = 2\left(\sum_{v \in V(H)} (d(v, w) + p)^\lambda + (1^\lambda + 2^\lambda + \dots + (p-1)^\lambda)\right) \\ + \sum_{v \in V(H')} (d(u, v) + 1)^\lambda.$$

Put  $A = (H_\lambda(w, G') + H_\lambda(v_p, G')) - (H_\lambda(w, G) + H_\lambda(v_p, G))$ , so we have:

$$A = (d_G(w) - 2) \sum_{v \in V(H')} (d(u, v) + p + 1)^\lambda + \sum_{v \in V(H)} (d(v, w) + p)^\lambda - p^\lambda \\ + (2 - d_G(w)) \sum_{v \in V(H')} (d(u, v) + 1)^\lambda - \sum_{v \in V(H)} d^\lambda(v, w) \\ = (d_G(w) - 2) \left( \sum_{v \in V(H')} ((d(u, v) + p + 1)^\lambda - (d(u, v) + 1)^\lambda) \right) \\ + \sum_{v \in V(H) - \{w\}} (d(v, w) + p)^\lambda + p^\lambda - p^\lambda - \sum_{v \in V(H)} d^\lambda(v, w) \\ = (d_G(w) - 2) \left( \sum_{v \in V(H')} ((d(u, v) + p + 1)^\lambda - (d(u, v) + 1)^\lambda) \right) \\ + \sum_{v \in V(H) - \{w\}} ((d(v, w) + p)^\lambda - d^\lambda(v, w)).$$

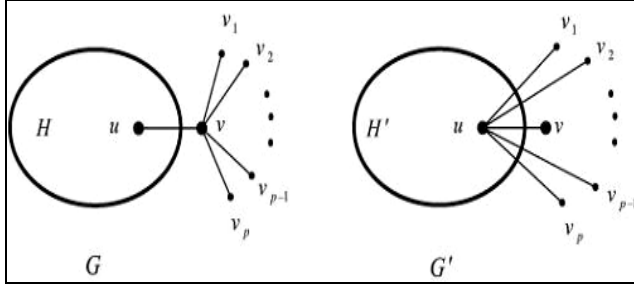
Since  $\lambda$  is a positive integer and  $d_G(w) > 2$ , thus  $A \geq 0$ . Let  $V(P) = \{v_1, v_2, \dots, v_p\}$ , then for every vertex  $x \in \{v_1, v_2, \dots, v_{p-1}\}$ , we have  $H_\lambda(x, G') = H_\lambda(x, G)$ . Now we assume  $x \in V(H) - \{w\}$ . So

$$H_\lambda(x, G) = d_G(x) \left( \sum_{v \in V(H)} d^\lambda(x, v) + \sum_{v \in V(H')} (d(u, v) + 1 + d(w, x))^\lambda \right. \\ \left. + (1 + d(w, x))^\lambda + \dots + (p + d(w, x))^\lambda \right), \\ H_\lambda(x, G') = d_G(x) \left( \sum_{v \in V(H)} d^\lambda(x, v) + \sum_{v \in V(H')} (d(u, v) + 1 + p + d(w, x))^\lambda \right. \\ \left. + (1 + d(w, x))^\lambda + \dots + (p + d(w, x))^\lambda \right).$$

Thus for every vertex  $x \in V(H) - \{w\}$ ,  $H_\lambda(x, G') \geq H_\lambda(x, G)$ . Now we assume  $x \in V(H')$ . So

$$H_\lambda(x, G) = d_G(x) \left( \sum_{v \in V(H')} d^\lambda(x, v) + \sum_{v \in V(H)} (d(w, v) + 1 + d(u, x))^\lambda \right. \\ \left. + (2 + d(u, x))^\lambda + \dots + (p + 1 + d(u, x))^\lambda \right), \\ H_\lambda(x, G') = d_G(x) \left( \sum_{v \in V(H')} d^\lambda(x, v) + \sum_{v \in V(H)} (d(w, v) + 1 + p + d(u, x))^\lambda \right. \\ \left. + (1 + d(u, x))^\lambda + \dots + (p + d(u, x))^\lambda \right).$$

Thus for every vertex  $x \in V(H')$ , we have  $H_\lambda(x, G') \geq H_\lambda(x, G)$ . Considering the above inequality, we obtain,  $H_\lambda(G') \geq H_\lambda(G)$  and the proof is completed.



**Figure 2.**  $\sigma$ -transform applied to  $G$  at vertex  $v$ .

Let  $v$  be a vertex of degree  $p+1$  in a graph  $G$ , which is not a star, such that  $vv_1, vv_2, \dots, vv_p$  are pendant edges incident with  $v$  and  $u$  is the neighbor of  $v$  distinct from  $v_1, v_2, \dots, v_p$ . We form a graph  $G' = \sigma(G, v)$  by removing edges  $vv_1, vv_2, \dots, vv_p$  and adding new edges  $uv_1, uv_2, \dots, uv_p$ . We say that  $G'$  is a  $\sigma$ -transform of  $G$  (see Figure 2). Note that if  $d_G(u) = 1$ , then  $G$  and  $G'$  are isomorphic [27].

**Theorem 2.2.** Let  $G' = \sigma(G, v)$  be a  $\sigma$ -transform of  $G$  and  $\lambda$  be a positive integer. Then  $H_\lambda(G) \geq H_\lambda(G')$ , and the equality is held if and only if  $d_G(u) = 1$ .

**Proof.** The only vertices that change the degree after performing the  $\sigma$ -transform are  $u$  and  $v$ . So

$$\begin{aligned} H_\lambda(u, G) &= d_G(u) \left( \sum_{x \in V(H)} d^\lambda(x, u) + 1 + 2^\lambda p \right), \\ H_\lambda(u, G') &= (d_G(u) + p) \left( \sum_{x \in V(H)} d^\lambda(x, u) + 1 + p \right), \\ H_\lambda(v, G) &= (p+1) \left( \sum_{x \in V(H)} (d(x, u) + 1)^\lambda + p \right), \\ H_\lambda(v, G') &= \sum_{x \in V(H)} (d(x, u) + 1)^\lambda + 2^\lambda p. \end{aligned}$$

Put  $A = (H_\lambda(u, G) + H_\lambda(v, G)) - (H_\lambda(u, G') + H_\lambda(v, G'))$ , thus

$$\begin{aligned}
 A &= p \sum_{x \in V(H)} (d(x, u) + 1)^\lambda + 2^\lambda p d_G(u) - p d_G(u) - p \sum_{x \in V(H)} d^\lambda(x, u) - 2^\lambda p \\
 &= p \left( \sum_{x \in V(H)} (d(x, u) + 1)^\lambda - \sum_{x \in V(H)} d^\lambda(x, u) \right) + 2^\lambda p (d_G(u) - 1) - p d_G(u).
 \end{aligned}$$

For every vertex  $x \in V(P) = \{v_1, v_2, \dots, v_p\}$ , we have:

$$\begin{aligned}
 H_\lambda(x, G) &= \sum_{x \in V(H)} (d(x, u) + 2)^\lambda + 1 + 2^\lambda (p - 1), \\
 H_\lambda(x, G') &= \sum_{x \in V(H)} (d(x, u) + 1)^\lambda + 2^\lambda p.
 \end{aligned}$$

Then

$$H_\lambda(x, G) - H_\lambda(x, G') = \sum_{x \in V(H)} ((d(x, u) + 2)^\lambda - (d(x, u) + 1)^\lambda) + 1 - 2^\lambda.$$

For every vertex  $x \in V(H) - \{u\}$ , we have:

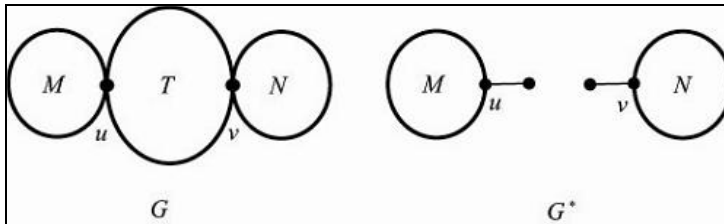
$$\begin{aligned}
 H_\lambda(x, G) &= d_G(x) \left( \sum_{y \in V(H)} d^\lambda(x, y) + (d(x, u) + 1)^\lambda + p(d(x, u) + 2)^\lambda \right), \\
 H_\lambda(x, G') &= d_G(x) \left( \sum_{y \in V(H)} d^\lambda(x, y) + (d(x, u) + 1)^\lambda + p(d(x, u) + 1)^\lambda \right).
 \end{aligned}$$

It is clear that for every vertex  $x \in V(H) - \{u\}$ ,  $H_\lambda(x, G) \geq H_\lambda(x, G')$ .

If  $d_G(u) = 1$ , then  $G \cong G'$ , so we assume that  $d_G(u) \geq 2$ . By considering the above inequality, we have,  $H_\lambda(G) \geq H_\lambda(G')$  and the proof is completed.

**Corollary 2.3.** Let  $T$  be a tree on  $n \geq 2$  vertices and  $\lambda$  be a positive integer. Then  $H_\lambda(S_n) < H_\lambda(T) < H_\lambda(P_n)$ .

**Proof.** Let  $T$  be a tree on  $n$  vertices. By using of Theorem 2.1, we easily get that  $H_\lambda(T) < H_\lambda(P_n)$  and by using of Theorem 2.2, we easily get that  $H_\lambda(S_n) < H_\lambda(T)$ .



**Figure 3.** The graphs  $G$  and  $G^*$ .

Let  $T$  be a tree on  $n \geq 2$  vertices. Then  $D'(S_n) < D'(T) < D'(P_n)$ . In the following Corollary, we will find a bound for  $H_\lambda$  in bicyclic graphs.

**Corollary 2.4.** Let  $G$  and  $G^*$  be two graphs which is shown in Figure 3, where  $M$  and  $N$  are vertex disjoint cycles,  $T$  is a tree with  $k \geq 3$  vertices,  $V(M) \cap V(T) = \{u\}$ ,  $V(N) \cap V(T) = \{v\}$ ,  $G^*$  is formed from  $G$  by setting the tree  $T$  to be  $P_k$  with end vertices  $u$  and  $v$ . Suppose that  $G \neq G^*$  and  $\lambda$  is a positive integer. If  $|V(M)|, |V(N)| \geq 2$ , then  $H_\lambda(G) < H_\lambda(G^*)$ .

**Proof.** The proof follows from Corollary 2.3.

**Lemma 2.5.** Let  $G$  be a connected graph with at least three vertices and  $\lambda$  be a negative integer.

- (i) If  $G$  is not isomorphic to  $K_n$ , then  $H_\lambda(G) < H_\lambda(G+e)$ , where  $e \in E(\overline{G})$ ,
- (ii) If  $G$  has an edge  $e$  not being a cut edge, then  $H_\lambda(G) > H_\lambda(G-e)$ .

**Proof.** (i) Suppose that  $G$  is not a complete graph. Then there exists a pair of vertices  $u$  and  $v$  in  $G$  such that  $uv \in E(\overline{G})$ . It is obvious that  $d_G(x, y) \geq d_{G+uv}(x, y)$  for any pair of vertices  $x$  and  $y$  in  $G$ . Also, we have  $d_G(u, v) > 1 = d_{G+uv}(u, v)$ . Moreover,  $d_{G+uv}(w) \geq d_G(w)$  for any  $w$  in  $G$ . Because  $\lambda$  is negative integer, we have  $d_{G+uv}^\lambda(u, v) > d_G^\lambda(u, v)$ , so  $H_\lambda(G) < H_\lambda(G+e)$ .

(ii) Since the edge  $e$  is not a cut edge in  $G$ , we have  $G-e$  is connected and not isomorphic to the complete graph of the same order. Thus, by (i), we have  $H_\lambda(G-e) < H_\lambda((G-e)+e) = H_\lambda(G)$ , as expected.

Let  $U_{n,k}$  be the set of all unicyclic graphs of order  $n \geq 3$  with girth  $k \geq 3$ . Also, let  $H_{n,k}$  be a subset of  $U_{n,k}$  such that contain a cycle  $C_k$  and the remaining vertices of graph make up only a subgraph that has exactly a common vertex with  $C_k$ . Obviously this subgraph is a tree. By  $L_{n,k}$ , we denote the graph obtained from  $C_k$  and  $P_{n-k+1}$  by indentifying a vertex of  $C_k$  with an end vertex of  $P_{n-k+1}$  and by  $S_{n,k}$  we denote the graph obtained from  $C_k$  and  $S_{n-k+1}$  by indentifying a vertex of  $C_k$  with a vertex of maximum degree of  $S_{n-k+1}$ . Now we characterize the minimum and maximum generalization degree distance over this special classes of unicyclic graphs.



**Theorem 2.6.** If  $G \in H_{n,k}$ , and  $\lambda$  is a positive integer, then  $H_\lambda(G) \leq H_\lambda(L_{n,k})$ , and equality is held if and only if  $G \cong L_{n,k}$ .

**Proof.** Since  $H_{n,k}$  contain only one cycle and one tree, according to Corollary 2.3, the maximum  $H_\lambda$  occur when tree is a path. If path is connected to cycle with vertex of minimum degree, by  $\pi$ -transform,  $H_\lambda$  is more than of the case that path connected to cycle with vertex of maximum degree. Therefore bound is obtained by calculating  $H_\lambda(L_{n,k})$  and the proof is completed.

**Theorem 2.7.** If  $G \in H_{n,k}$ , and  $\lambda$  is a positive integer. Then  $H_\lambda(G) \geq H_\lambda(S_{n,k})$ , and equality is held if and only if  $G \cong S_{n,k}$ .

**Proof.** Since  $H_{n,k}$  contain only one cycle and one tree, according to Corollary 2.3, the minimum  $H_\lambda$  occur when tree is a star. If star is connected to cycle with vertex of maximum degree, by  $\sigma$ -transform  $H_\lambda$  is less than of the case that star connected to cycle with vertex of minimum degree. Therefore bound is obtained by calculating  $H_\lambda(S_{n,k})$  and the proof is completed.

Let  $\lambda$  be a negative integer. Define  $D^{\lambda*}(u) = \sum_{v \in V(G) - \{u\}} d^\lambda(u, v)$ .

**Theorem 2.8.** Let  $G$  be a connected graph of order  $n \geq 2$  and size  $m \geq 1$  and  $\lambda$  is negative integer. Then

$M_1(G) + 2d^\lambda mn - d^\lambda M_1(G) - 2d^\lambda m \leq H_\lambda(G) \leq (1 - 2^\lambda)M_1(G) + 2^{\lambda+1}mn - 2^{\lambda+1}m$  and equality is held if and only if  $d \leq 2$ , where  $d$  is the diameter of  $G$ .

**Proof.** First, let us prove that the right-hand side inequality holds. For each vertex  $x$  in  $G$ , we have

$$\begin{aligned} D^{\lambda*}(x) &= d_G(x) + \sum_{y \in V(G) - N[x]} d_G^\lambda(x, y) \\ &\leq d_G(x) + 2^\lambda(n - d_G(x) - 1), \end{aligned}$$

where the equality is attained if and only if  $\varepsilon(x) \leq 2$ . So

$$\begin{aligned} H_\lambda(G) &= \sum_{x \in V(G)} d_G(x) D^{\lambda*}(x) \\ &\leq \sum_{x \in V(G)} d_G(x) (d_G(x) + 2^\lambda(n - d_G(x) - 1)) \\ &= (1 - 2^\lambda)M_1(G) + 2^{\lambda+1}mn - 2^{\lambda+1}m, \end{aligned}$$

where the equality is attained if and only if for each  $x$ ,  $\varepsilon(x) \leq 2$ . So,  $H_\lambda(G) \leq (1 - 2^\lambda)M_1(G) + 2^{\lambda+1}mn - 2^{\lambda+1}m$  with equality if and only if the diameter of  $G$  is at most 2, as desired. Now, we turn to the left-hand side inequality. For each vertex  $x$  in  $G$ ,

$$\begin{aligned} D^{\lambda*}(x) &= d_G(x) + \sum_{y \in V(G) - N[x]} d_G^\lambda(x, y) \\ &\geq d_G(x) + d^\lambda(n - d_G(x) - 1), \end{aligned}$$

where the equality is attained if and only if for any  $y \in V(G) - N[x]$ ,  $d(x, y) = d$ , implying that  $d \leq 2$ . Therefore,

$$\begin{aligned} H_\lambda(G) &= \sum_{x \in V(G)} d_G(x) D^{\lambda*}(x) \\ &\geq \sum_{x \in V(G)} d_G(x) (d_G(x) + d^\lambda(n - d_G(x) - 1)) \\ &= M_1(G) + 2d^\lambda mn - d^\lambda M_1(G) - 2d^\lambda m, \end{aligned}$$

where the equality is attained if and only if  $d \leq 2$ . This completes the proof.

A cactus is a connected graph each of whose blocks is either a cycle or an edge. If a cactus has no cycles, then it is just a tree, and if it has exactly a cycle, then it is a unicyclic graph. For  $0 \leq k \leq \frac{n-1}{2}$ , we let  $G_n^k$  be an  $n$ -vertex  $k$ -cycle cactus obtained from the  $n$ -vertex star by adding  $k$  independent edges among  $n-1$  pendant vertices.

The following Lemma is a result of [28].

**Lemma 2.9.** Let  $G$  be an  $n$ -vertex  $k$ -cycle cactus with  $0 \leq k \leq \frac{n-1}{2}$ .

Then  $M_1(G) \leq n^2 - n + 6k$ , and equality is held if and only if  $G \cong G_n^k$ .

**Theorem 2.10.** Let  $G$  be an  $n$ -vertex  $k$ -cycle cactus with  $0 \leq k \leq \frac{n-1}{2}$

and  $\lambda$  be a negative integer. Then we have:

$$H_\lambda(G) \leq (1 - 2^\lambda)(n^2 - n + 6k) + 2^{\lambda+1}(n^2 - 2n + kn + 1 - k),$$

and equality is held if and only if  $G \cong G_n^k$ .

**Proof.** Note that  $G$  has  $n+k-1$  edges. By Theorem 2.8 and Lemma 2.9, we have

$$H_\lambda(G) \leq (1 - 2^\lambda)M_1(G) + 2^{\lambda+1}mn - 2^{\lambda+1}m$$

$$\begin{aligned} &\leq (1-2^\lambda)(n^2 - n + 6k) + 2^{\lambda+1}(n-1)(n+k-1) \\ &= (1-2^\lambda)(n^2 - n + 6k) + 2^{\lambda+1}(n^2 - 2n + kn + 1 - k). \end{aligned}$$

The equality is held if and only if the diameter of  $G$  is 2 and  $G \cong G_n^k$ . Note that  $G_n^k$  has diameter 2. Thus,

$$H_\lambda(G) \leq (1-2^\lambda)(n^2 - n + 6k) + 2^{\lambda+1}(n^2 - 2n + kn + 1 - k)$$

with equality if and only if  $G \cong G_n^k$ , so completing the proof.

By Theorem 2.8, we immediately have the following results for  $H_\lambda$  of trees and unicyclic graphs, respectively.

**Corollary 2.11.** Let  $G$  be a unicyclic graph on  $n \geq 3$  vertices and  $\lambda$  be a negative integer. Then  $H_\lambda(G) \leq (1-2^\lambda)(n^2 - n + 6) + 2^{\lambda+1}(n^2 - n)$  and equality is held if and only if  $G \cong G_n^1$ .

The final result in this paper is related to trees and we obtain an upper and lower bound for generalization degree distance of trees.

**Corollary 2.12.** Let  $T$  be a tree on  $n \geq 2$  vertices and  $\lambda$  be a negative integer. Then  $H_\lambda(P_n) < H_\lambda(T) \leq (1-2^\lambda)(n^2 - n) + 2^{\lambda+1}(n^2 - 2n + 1)$ , and the right equality is held if and only if the tree is a star,  $T \cong S_n$ .

**Proof.** By Theorem 2.8, the right-hand side inequality is held. Now, we turn to the left-hand side inequality. We prove among all nontrivial connected graphs of order  $n$ , the graphs with the maximum and minimum  $H_\lambda$  are  $K_n$  and  $P_n$ , respectively. The case of  $n = 2$  is trivial. So we suppose that  $n \geq 3$  and we first prove that  $K_n$  is maximal with respect to  $H_\lambda$ . If  $G$  is not a complete graph, then we can add some edges into  $G$  such that we obtain  $G \cong K_n$ . By Lemma 2.5,  $H_\lambda(G) \leq H_\lambda(K_n)$ , with equality if and only if  $G \cong K_n$ . Now, let us prove that  $P_n$  is minimal with respect to  $H_\lambda$ . Suppose first that  $G$  is not isomorphic to a tree. Let  $T(G)$  be a spanning tree of  $G$ . It then follows from Lemma 2.5, that  $H_\lambda(G) > H_\lambda(T(G))$ . So we need only to consider the case of  $G$  is a tree. If  $G$  is not isomorphic to the path, then by using the  $\pi$ -transform on  $G$  for some times according to the status of graph  $G$  we obtain the path  $P_n$ . Then by Theorem 2.1 and negative integer  $\lambda$ , we have  $H_\lambda(P_n) < H_\lambda(G)$ , as expected.

By the above Corollary, if  $T$  is a tree on  $n \geq 2$  vertices, then  $H_\lambda(P_n) < H_\lambda(T) \leq H_\lambda(S_n)$ .

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